Identification of critical node clusters for consensus in large-scale spatial networks

Vishaal Krishnan * **Sonia Martínez** *

* *Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla CA 92093 USA*
{ v6krishn; soniamd } @ucsd.edu.

Abstract: In this work, we address the problem of identifying a set of nodes that are critical to the rate of convergence of consensus dynamics in large-scale spatial networks. By assuming that nodes are uniformly distributed over a spatial region and that these can communicate with others in an infinitesimally small neighborhood, we start by formulating the consensus problem by means of a partial differential equation involving the Laplace operator, subject to Neumann boundary condition. As with its finite-dimensional counterpart, we observe how the performance of these dynamics is directly related to the second smallest eigenvalue of the Laplace operator over the domain of interest. We then reduce the critical node set identification problem to that of finding a ball of fixed radius, whose removal minimizes the rate of convergence over the residual domain. This leads us to consider two functional optimization problems. First, we treat the problem of determining the second smallest eigenvalue for a fixed domain by minimizing an energy functional. We characterize the critical points of the energy functional, and then construct the gradient dynamics that converge to the set of critical points. We then prove that the only locally asymptotically stable critical point is the second eigenfunction of the Laplace operator. Building on these results, we consider the critical ball identification problem, provide a characterization of the critical points, and define gradient dynamics to converge asymptotically to these points.

Keywords: Large scale networks, critical node clusters, Laplacian consensus

1. INTRODUCTION

With novel advances in low-cost sensing, communication, and computational systems, large-scale spatial networks are becoming increasingly pervasive. Much work has been devoted to the analysis and design of networks composed of a small number of agents; however, the available results do not translate well to large-scale groups. In a large-scale system, we are generally concerned with questions related to the macroscopic performance of the network. Thus, specifying the global state of the network through the states of individual nodes is needlessly complicated. Guaranteeing the performance of each individual agent is also purposeless, as failures in a small subset of agents should be both acceptable (due to high redundancy) and unavoidable. This highlights the need to adopt alternative modeling frameworks and develop new control-theoretical tools to account for the characteristic (spatial) properties of large-scale networked systems. Such a framework could also be beneficial to solving design problems such as the selection of nodes in a network for improved controllability, robustness, or observability, objectives which routinely result in hard combinatorial optimization problems. Motivated by this, we adopt here a continuum modeling framework for large-scale spatial networks, and study the question of critical node-set selection for a spatial consensus problem.

The general problem of optimizing a performance metric with respect to a network design parameter has received much attention. One class of problems, to which the current work belongs, is that of critical node identification. This is concerned with identifying those nodes in the network whose removal results in the maximum deterioration of network performance with respect to a chosen metric. Investigations adopting various notions of criticality exist in the literature (Ventresca and Aleman, 2015). Although the word "critical" has been used in several contexts, the theoretical problems that underlie these investigations are not necessarily similar, and in many cases, vastly different.

Other related problems include the leader selection problem, where a given metric is optimized by the choice of a leader node or a set of leader nodes. This has been explored in a variety of contexts, for improved network controllability (Clark et al., 2012; Ji et al., 2006; Commault and Dion, 2013), and to minimize mean-square deviation from consensus in the presence of noise (Lin et al., 2014; Patterson and Bamieh, 2010), to name a few. Yet another is the optimal edge-weight selection problem, where a cost is optimized by tuning edge weights. The problem of improving the rate of convergence for consensus dynamics by edge-weight selection has been studied in (Xiao and Boyd, 2004; Boyd, 2006; Hao and Barooah, 2012).

Prior works can be found in the literature on approximating large networks by weighted graphs with an underlying set of continuum cardinality. The book (Lovász, 2012) contains a treatment of such an approach, where large networks are approximated by limit objects, called graphons, of convergent sequences of large dense graphs (under appropriately defined notions of distance between graphs and convergence). In this approach, the nodes in the network are usually indexed by the unit interval $[0,1]$. Extending this idea to spatial networks, where the nodes are embedded in a domain $\Omega \in \mathbb{R}^N$ in physical space, the nodes can be thought to be indexed by their positions $x \in \Omega$. Combining these notions in the context of network consensus dynamics, the objects of interest are the Laplacian of the graph, and its continuum counterpart, the Laplace operator on the domain. Theoretical results concerning the convergence of the graph Laplacian to the Laplace operator can be found in (Belkin and Niyogi, 2005) and (Belkin and Niyogi, 2007). These also motivate the formulation adopted in this paper to study large-scale spatial networks.

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There have been severals attempts to investigate problems linking the shape of a domain with the sequence of eigenvalues of the Laplace operator, for various boundary conditions, although, to the best of our knowledge, there is no general result on the critical subset identification for general domains. The book (Henrot, 2006) contains an overview of the literature on extremum problems for eigenvalues of elliptic (e.g. Laplace) operators.

In this paper, we are interested in the problem of identifying critical nodes for network consensus, nodes whose removal leads to maximum deterioration in the rate of convergence for the residual network. We make a continuum approximation to model the consensus dynamics by the heat equation with Neumann boundary condition, where the rate of convergence of solutions to steady state is governed by the smallest positive (also the second smallest) eigenvalue of the Laplace operator. It is worth noting that in the finite dimensional counterpart, the second smallest eigenvalue of the Laplacian matrix of a graph plays a special role, it denotes the algebraic connectivity of the graph and is significant in many respects. We formulate the critical node set identification problem in large-scale spatial networks as a hole placement problem, such that the optimal placement of the hole in the domain minimizes the smallest positive eigenvalue of the Laplace operator for the residual domain. The problem of optimizing the (smallest positive) eigenvalue with respect to hole position presents some inherent difficulties in that it is non-convex, thereby limiting our goal at the outset to achieving convergence to a local optimum. We consider two functional optimization problems. First, we treat the problem of determining the second smallest eigenvalue for a fixed domain. This is achieved by minimizing an energy functional (given by the Min-max theorem), characterize the critical points of the energy functional, and construct the gradient dynamics in a Banach space that converge to the set of critical points. We then prove that the only locally asymptotically stable critical point for the dynamics is the second eigenfunction of the Laplace operator. Building on these results, we consider the optimal hole placement problem, provide a characterization of the critical points in the interior of the domain, and define gradient dynamics to converge asymptotically to these points.

This paper is organized as follows. Section 3 introduces the preliminaries of the functional form of Laplacian consensus dynamics. The optimization problem is formulated in Section 4, followed by the main analysis in Section 5. We then conclude with the summary and future directions in Section 7.

2. NOTATION

In this section, we present some of the notation used in the rest of the paper. We denote by $\mathbf{1}_n$ the vector of ones $\mathbf{1}_n^\top = (1,\ldots,1)^\top \in \mathbb{R}^n$. We denote by $B_r(x)$ the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^N$, and by $|\Omega|$ the Lebesgue measure of the set $\Omega \subset \mathbb{R}^N$. The set of square-integrable functions on Ω is represented by $L^2(\Omega)$. In other words, $L^2(\Omega) = \{f : \Omega \to \mathbb{R} \mid \int_\Omega |f|^2 dv < \infty\}$, where dv is the standard Lebesgue measure. By extension, $L^2(\Omega,\rho)$ represents the set of square-integrable functions with respect to the measure ρdv, where ρ is a positive real-valued function on Ω. When clear from the context, and with a slight abuse of notation, we will denote $\int_\Omega f dv$ simply as $\int_\Omega f$, for some $f \in L^2(\Omega)$. For $f,g \in L^2(\Omega)$ (resp. $f,g \in L^2(\Omega,\rho)$), we let $\langle f,g\rangle = \int_\Omega fg dv$ (resp. $\langle f,g\rangle_{L^2(\Omega,\rho)} = \int_\Omega fg\rho dv$) denote the inner product and $\|f\|^2 = \langle f,f\rangle$ denote the corresponding induced norm. Finally, we denote by $H^1(\Omega) = \{f \in L^2(\Omega) \mid \int_\Omega |\nabla f|^2 dv < \infty\}$.

3. LAPLACIAN CONSENSUS IN FUNCTIONAL FORM

In this section, we present some preliminaries on the Laplacian consensus problem in functional form. We start introducing the Laplacian consensus dynamics in the discrete setting (on a graph), and then use a convergence result from (Belkin and Niyogi, 2005) in the limit as $n \to \infty$ to approximate the Laplacian matrix by the Laplace operator on the spatial domain (Evans, 1998).

Let $\mathscr{V} = \{1,\ldots,n\}$ be a set of n agents, which are distributed uniformly over a bounded open set $\Omega \subset \mathbb{R}^N$ (with C^1-boundary $\partial\Omega$) and with positions $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\} \subseteq \Omega$. We define a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}_{\text{disk}}(h))$ over the set of agents, for some $h \in \mathbb{R}_{>0}$, where $\mathscr{E}_{\text{disk}}(h)$ is the set of undirected edges defined as follows: $(i,j) \in \mathscr{E}_{\text{disk}}(h)$ if and only if $|\mathbf{x}_i - \mathbf{x}_j| < h$. Let L_n^h be the symmetric graph Laplacian matrix associated with $\mathscr{G}$. That is, $L_n^h = D_n^h - A_n^h$, where A_n^h is the adjacency matrix associated with $\mathscr{G}$ and D_n^h is the diagonal degree-matrix (Bullo et al., 2009). The adjacency matrix A_n^h can thus be thought to be extracted from a uniform kernel K_h as $(A_n^h)_{ij} = K_h(|\mathbf{x}_i - \mathbf{x}_j|^2)$. We further assume that the uniform kernel above is approximated by an appropriate smooth kernel. The continuous-time Laplacian consensus on a function $\phi^d : \mathbb{R} \to \mathbb{R}^n$ is given by:

$$\frac{d}{dt}\phi^d = -L_n^h \phi^d. \tag{1}$$

As $n \to \infty$ and $h \to 0$, the graph Laplacian L_n^h converges to the Laplace operator $\mathscr{L}$ in Ω (Belkin and Niyogi, 2005; Singer, 2006), where:

$$\mathscr{L}u = -\Delta u. \tag{2}$$

In this way, given a functional (continuum) approximation of ϕ^d, in the form of $\phi : \mathbb{R} \times \Omega \to \mathbb{R}$, we obtain the functional form of the consensus dynamics as follows:

$$\partial_t \phi = -\mathscr{L}\phi = \Delta\phi. \tag{3}$$

This PDE is accompanied by an additional *Neumann boundary condition* motivated as follows. Let $m^d = \sum_{i=1}^n \phi_i^d = \mathbf{1}_n^\top \phi^d$ be the sum of the consensus variable across a discrete set of agents. From (1), we have:

$$\frac{d}{dt}m^d = \mathbf{1}_n^\top \frac{d}{dt}\phi^d = -\mathbf{1}_n^\top L_n^r \phi^d = 0,$$

since $\mathbf{1}_n \in \ker(L_n^r)$. Equivalently, in the functional setting, the sum m^d converges (as $n \to \infty$ and $h \to 0$) to the integral $m = \int_\Omega \phi \; d\nu$. From (2) and (3), and using the Divergence Theorem (Evans, 1998), we have:

$$\partial_t m = \int_\Omega \partial_t \phi \; d\nu = \int_\Omega \Delta\phi \; d\nu = \int_{\partial\Omega} \nabla\phi \cdot \mathbf{n} \; dS.$$

Thus, we impose the Neumann boundary condition on the boundary $\partial\Omega$ for the functional ϕ to conserve m (s.t. $\partial_t m = 0$):

$$\nabla\phi \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \tag{4}$$

In other words, the Neumann boundary condition imposes the constraint that the flux across the boundary is zero.

For analysis purposes, consider the energy function of the discrete-agent setting for the consensus problem:

$$E^d = \frac{1}{2}\sum_i \sum_j (\phi_i^d - \phi_j^d)^2 = \frac{1}{2}\langle \phi^d, L_n^h \phi^d \rangle.$$

In the functional setting, the corresponding energy functional takes the form:

$$E = \frac{1}{2}\langle \phi, \mathscr{L}\phi \rangle_{L^2(\Omega)}.$$

The time derivative of E under the boundary condition (4) can be computed to be:

$$\begin{aligned} \frac{d}{dt}E &= \frac{1}{2}\partial_t \left(\int_\Omega \phi \mathscr{L}\phi \; d\nu \right) = \frac{1}{2}\partial_t \left(\int_\Omega |\nabla\phi|^2 d\nu \right) \\ &= \int_\Omega \nabla\phi \cdot \nabla(\partial_t \phi) d\nu = -\int_\Omega |\mathscr{L}\phi|^2 d\nu \\ &= -\langle \mathscr{L}\phi, \mathscr{L}\phi \rangle_{L^2(\Omega)}. \end{aligned} \tag{5}$$

Note that the operator $\mathscr{L}$ is elliptic. From the theory of elliptic partial differential operators (Evans, 1998), for the Neumann problem, the operator $\mathscr{L}$ has an infinite sequence of eigenvalues $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_m \le \ldots$, whose corresponding eigenfunctions $\{\psi_i\}_{i=1}^\infty$ form an orthonormal basis for $L^2(\Omega)$.

Remark 1. *The Laplace operator $\mathscr{L}$ with Neumann boundary condition is self-adjoint, which implies that the algebraic multiplicities of the eigenvalues are equal to their geometric multiplicities.* •

Therefore, we can express $\phi(t,\mathbf{x})$ as:

$$\phi(t,\mathbf{x}) = \sum_i c_i(t)\psi_i(\mathbf{x}).$$

Now, we obtain:

$$\mathscr{L}\phi(t,\mathbf{x}) = \sum_i c_i(t)\mu_i\psi_i(\mathbf{x}).$$

Substituting in (5) and from the ordering of eigenvalues, we obtain the relation:

$$\frac{d}{dt}E \le -2\mu_2 E,$$

which implies that $E(t) \le e^{-2\mu_2 t}E(0)$. We see that the second eigenvalue $\mu_2(\mathscr{L}(\Omega)) = \mu_2$ governs the convergence rate of the functional consensus dynamics.

Using the Min-max theorem (Evans, 1998) for the operator $\mathscr{L}$, one can determine:

$$\mu_2(\mathscr{L}(\Omega)) = \inf_{\psi \in \{\psi_1\}^\perp} \frac{\langle \psi, \mathscr{L}\psi \rangle_{L^2(\Omega)}}{\langle \psi, \psi \rangle_{L^2(\Omega)}}, \tag{6}$$

where $\{\psi_1\}^\perp = \{\psi \in H^1(\Omega) \,|\, \psi \neq 0, \; \int_\Omega \psi_1 \psi \; d\nu = 0\}$, and ψ_1 is constant, the eigenfunction corresponding to $\mu_1 = 0$. This implies that $\{\psi_1\}^\perp = \{\psi \in H^1(\Omega) \,|\, \int_\Omega \psi \; d\nu = 0\}$. Therefore, using the Divergence theorem and applying the Neumann boundary condition, Equation (6) becomes:

$$\mu_2(\mathscr{L}(\Omega)) = \inf_{\substack{\int_\Omega \psi d\nu = 0, \\ \psi \neq 0}} \frac{\int_\Omega |\nabla\psi|^2 d\nu}{\int_\Omega |\psi|^2 d\nu}.$$

The above can be reformulated as:

$$\mu_2(\Delta(\Omega)) = \inf_{\substack{\int_\Omega \psi d\nu = 0, \\ \int_\Omega |\psi|^2 d\nu = 1}} \int_\Omega |\nabla\psi|^2 d\nu. \tag{7}$$

4. PROBLEM FORMULATION

Here, we define the notion of criticality that we adopt in this paper. Critical agents are those located in Ω, whose removal results in the minimum rate of convergence for the residual network. In other words, these are the agents whose removal will affect more adversely the rate of convergence of consensus dynamics, making them the most valuable agents to be protected.

More precisely, this amounts to identifying a set $K \subset \Omega$ of given measure $|K| = c > 0$ such that $\mu_2(\Delta(\Omega \setminus K))$ is an infimum.

For a fixed K, from Equation (7), again with a Neumann boundary condition on ∂K, we obtain the following minimization problem over ψ:

$$\mu_2(\Delta(\Omega \setminus K)) = \inf_{\substack{\int_{\Omega\setminus K} \psi d\nu = 0, \\ \int_{\Omega\setminus K} |\psi|^2 d\nu = 1}} \int_{\Omega\setminus K} |\nabla\psi|^2 d\nu.$$

Further, the problem of identifying the critical nodes K^* can be formulated as:

$$K^* = \arg \inf_{\substack{K\subset\Omega, \\ |K| = c}} \inf_{\substack{\int_{\Omega\setminus K} \psi d\nu = 0, \\ \int_{\Omega\setminus K} |\psi|^2 d\nu = 1}} \int_{\Omega\setminus K} |\nabla\psi|^2 d\nu.$$

We restrict the search to a class of subsets $K = B_r(x) = \{y \in \Omega \,|\, |y - x| < r\} \subset \Omega$, open balls of radius r (such that $|B_r(x)| = c$). This reduces the search space to $\tilde{\Omega}_r = \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > r\}$, and the problem is reformulated as:

$$x^* = \arg \inf_{x \in \tilde{\Omega}_r} \inf_{\substack{\int_{\Omega\setminus B_r(x)} \psi d\nu = 0, \\ \int_{\Omega\setminus B_r(x)} |\psi|^2 d\nu = 1}} \int_{\Omega\setminus B_r(x)} |\nabla\psi|^2 d\nu. \tag{8}$$

Note that, in this setting, we assume that our domain Ω is such that there is no ball of radius r whose removal leads to a disconnection of Ω into several connected components. In other words, we assume that the residual set stays connected.

5. FUNCTIONAL OPTIMIZATION TO DETERMINE THE MOST CRITICAL NODES

In this section, we present the algorithm to determine the most critical nodes in the network, in a functional optimization framework. First, we begin with the analysis of the eigenvalue problem (7) (the inner optimization problem in (8)) for a fixed D. Its understanding will help us build a gradient dynamics that can be employed to solve the full critical node identification problem (8). In what follows, the proofs of the results are omitted and will appear in a forthcoming publication.

5.1 Eigenvalue determination problem for a fixed domain

We first consider the eigenvalue problem (7), characterize the critical points, construct and analyze a projected gradient dynamics to converge to the infimum. We write the optimization problem (for the smallest positive eigenvalue of the Laplace operator on a domain D with a C^1, Lipschitz boundary) as:

$$\inf_{\psi \in H^1(D)} \int_D |\nabla\psi|^2,$$
$$\text{s.t} \quad \int_D |\psi|^2 = 1, \quad \int_D \psi = 0,$$
$$\nabla\psi \cdot \mathbf{n} = 0 \text{ on } \partial D.$$

Let $\mathscr{S}_D = \{\psi \in H^1(D) \,|\, \int_D |\psi|^2 = 1, \int_D \psi = 0, \nabla\psi\cdot\mathbf{n} = 0 \text{ on } \partial D\}$. With this, we can rewrite the above problem as follows:

$$\inf_{\psi\in\mathscr{S}_D} \int_D |\nabla\psi|^2.$$

Critical points Here, we analyze the critical points ψ^* of the objective functional $J(\psi) = \int_D |\nabla\psi|^2$ in $\mathscr{S}_D$. Let $\delta\psi \in H^1(D)$ be a perturbation of ψ^*. The first variation of the Lagrangian $L(\psi,\mu,\lambda) = J(\psi) + \mu\left(1 - \int_D |\psi|^2\right) + \lambda \int_D \psi$, at a critical point ψ^* is zero (where $\int_D |\psi|^2 = 1$ and $\int_D \psi = 0$ are the constraints, as $\psi \in \mathscr{S}_D$ and the Neumann boundary condition is assumed implicitly.) Thus, $\left\langle \frac{\delta L}{\delta\psi}, \delta\psi \right\rangle (\psi^*,\mu^*,\lambda^*) = 2\int_D \nabla\psi^*\cdot\nabla(\delta\psi) - 2\mu^*\int_D \psi^*\delta\psi + \lambda^*\int_D \delta\psi = -2\int_D(\Delta\psi^* + \mu^*\psi^* - \frac{1}{2}\lambda^*)\ \delta\psi = 0$, for any $\delta\psi$ (note that the Neumann boundary condition was used in obtaining the equation.) Additionally, we also have $\left\langle \frac{\partial L}{\partial\mu}, \delta\mu \right\rangle (\psi^*,\mu^*,\lambda^*) = 1 - \int_D |\psi^*|^2 = 0$, and $\left\langle \frac{\partial L}{\partial\lambda}, \delta\lambda \right\rangle (\psi^*,\mu^*,\lambda^*) = \int_D \psi^* = 0$. Thus, the critical points of the objective functional $\psi^* \in \mathscr{S}_D$ are characterized by:

$$\Delta\psi^* + \mu^*\psi^* - \frac{1}{2}\lambda^* = 0.$$

Integrating the previous equation over D and using the Neumann boundary condition, we obtain $\lambda^* = 0$. Therefore, the critical points ψ^* satisfy:

$$\Delta\psi^* + \mu^*\psi^* = 0. \tag{9}$$

We now have the following lemma:

Lemma 1. *Of all the critical points ψ^* of the functional $J(\psi)$, the second eigenfunction ψ_2 of $\Delta(D)$ is the only minimizer of $J(\psi)$ in $\mathscr{S}_D$.* •

Projected gradient dynamics We now provide dynamics for convergence to the minimum value of $J(\psi)$ in $\mathscr{S}_D$. For smooth one-parameter families of functions $\{\psi(t,x)\}_{t\in\mathbb{R}_{\geq 0}}$ (with $x \in D$), the derivative of the objective functional J is given by:

$$\frac{dJ}{dt} = \left\langle \frac{\delta J}{\delta\psi}, \partial_t\psi \right\rangle = 2\int_D \nabla\psi\cdot\nabla(\partial_t\psi) = -2\int_D \partial_t\psi(\Delta\psi).$$

Thus, a gradient-descent dynamics is:

$$\partial_t\psi = -\frac{1}{2}\frac{\delta L}{\delta\psi} = \Delta\psi.$$

Now, we project this primal dynamics onto the tangent space of the set $\mathscr{S}_D$. For $\psi \in \mathscr{S}_D$, we require that $\langle\psi,\partial_t\psi\rangle = 0$ and $\int_D \partial_t\psi = 0$, which are satisfied if (this will be shown through Lemma 2):

$$\partial_t\psi = \Delta\psi - \frac{\langle\Delta\psi,\psi\rangle}{\|\psi\|^2}\psi = \Delta\psi - \langle\Delta\psi,\psi\rangle\,\psi,$$

since $\|\psi\| = 1$ for $\psi \in \mathscr{S}_D$. Further, using $J(\psi) = -\langle\Delta\psi,\psi\rangle$, we get the projected gradient dynamics:

$$\partial_t\psi = \Delta\psi + J(\psi)\psi. \tag{10}$$

Lemma 2. *The set $\mathscr{S}_D$ is invariant with respect to the projected gradient dynamics* (10). •

The equilibria ψ^* of (10) are given by:

$$\Delta\psi^* + J(\psi^*)\psi^* = 0,$$

and ψ^* also satisfies the Neumann boundary condition $\nabla\psi^* = 0$ on ∂D. Clearly, $J(\psi^*)$ is an eigenvalue, and so let $\mu^* = J(\psi^*)$. It is also clear that the equilibria of the projected gradient dynamics are also the critical points of the functional J over the set $\mathscr{S}_D$. We now have the following convergence result for solutions to the projected gradient dynamics.

Lemma 3. *The solutions to the projected gradient dynamics* (10) *in $\mathscr{S}_D$ converge in the L^2-norm to the set of equilibria of* (10). •

We now have the following stability result for the equilibria of the linearized projected gradient dynamics (10).

Lemma 4. *The second eigenfunction ψ_2 is the only locally asymptotically stable equilibrium in $\mathscr{S}_D$ for the projected gradient dynamics.* •

5.2 Critical node set identification

We now consider the full optimization problem:

$$\begin{aligned} x^* &= \arg\inf_{x\in\tilde{\Omega}_r} \mu_2(x) \\ &= \arg\inf_{x\in\tilde{\Omega}_r} \inf_{\substack{\int_{\Omega\setminus B_r(x)}\psi=0,\\ \int_{\Omega\setminus B_r(x)}|\psi|^2 d\nu=1}} \int_{\Omega\setminus B_r(x)} |\nabla\psi|^2 d\nu. \end{aligned} \tag{11}$$

We will assume that the second eigenvalue μ_2 is simple for the domains under consideration. Eigenvalues are differentiable with respect to domain perturbations for domains with Lipschitz boundaries (Henrot, 2006).

Critical points We now characterize the critical points of the functional μ_2 in the interior of the domain.

Lemma 5. *The first-order condition for a critical point x^* of the functional μ_2 in the interior of the domain is given by:*

$$\mu_2^*\left(\int_{\partial B_r(x^*)} |\psi_2^*|^2\mathbf{n}\right) = \int_{\partial B_r(x^*)} |\nabla\psi_2^*|^2\mathbf{n}, \tag{12}$$

where $(\mu_2^* = \mu_2(x^*), \psi_2^*)$ is the second eigenpair and **n** *is the outward normal to $\partial B_r(x^*)$.* •

Projected gradient dynamics We now construct the dynamics to converge to a critical point of μ_2 in the interior of the domain. Note that the function $\mu_2(x)$ is not known explicitly for a general domain $\Omega\setminus B_r(x)$. We reformulate the optimization problem (8) as:

$$\begin{aligned} x^* = \arg\inf_{x\in\tilde{\Omega}_r} \mu_2(x) &= \arg\inf_{x\in\tilde{\Omega}_r} \inf_{\substack{\int_{\Omega\setminus B_r(x)}\psi=0,\\ \int_{\Omega\setminus B_r(x)}|\psi|^2=1}} \int_{\Omega\setminus B_r(x)} |\nabla\psi|^2 d\nu \\ &= \arg_1 \inf_{(x,\psi)\in\tilde{\Omega}\times\Psi(x)} \int_{\Omega\setminus B_r(x)} |\nabla\psi|^2 d\nu, \end{aligned} \tag{13}$$

where the set $\Psi(x)$ is defined as:

$$\Psi(x) = \{\psi \in H^1(\Omega \setminus B_r(x)) \mid \int_{\Omega\setminus B_r(x)} \psi = 0, \int_{\Omega\setminus B_r(x)} |\psi|^2 = 1\}. \tag{14}$$

and $\arg_1$ indicates the first argument x in (x, ψ). We also define the set $\Psi = \cup_{x\in\tilde{\Omega}_r} \Psi(x)$.

Let $\{x(t)\}_{t\in\mathbb{R}_{\geq 0}}$ be a smooth curve in $\tilde{\Omega}_r$ and $\{\psi(t,y)\}_{t\in\mathbb{R}_{\geq 0}}$ (with $y \in \Omega \setminus B_r(x(t))$,) a smooth one-parameter family of functions on $\Omega \setminus B_r(x(t))$. We now consider the following dynamics:

$$\begin{aligned}
&\frac{dx}{dt} = \mathbf{v} = \begin{cases} \mathbf{v}_{int}, \ x \in \text{int}\,\tilde{\Omega}_r \\ \mathbf{v}_{int} - (\mathbf{v}_{int}\cdot\tilde{\mathbf{n}})\tilde{\mathbf{n}}, \ x \in \partial\tilde{\Omega}_r \end{cases}\\
&\mathbf{v}_{int} = \int_{\partial B_r(x)} |\nabla\psi|^2 \mathbf{n} - J(\psi)\int_{\partial B_r(x)} |\psi|^2 \mathbf{n},\\
&\partial_t \psi = \Delta\psi + J(\psi)\psi + a\psi + b,\\
&\nabla\psi\cdot\mathbf{n} = 0, \quad \text{on } \partial\Omega \cup \partial B_r(x),
\end{aligned} \tag{15}$$

where $\mathbf{n}$ and $\tilde{\mathbf{n}}$ denote the outward normals to $\partial B_r(x)$ and $\partial\tilde{\Omega}_r$ respectively, $a = -\frac{1}{2}\mathbf{v}\cdot\left(\int_{\partial B_r(x)} |\psi|^2\,\mathbf{n}\right)$ and $b = -\frac{1}{|\Omega|-c}\mathbf{v}\cdot\left(\int_{\partial B_r(x)} \psi\,\mathbf{n}\right)$, with $c = |B_r(x)|$, for all $x \in \tilde{\Omega}_r$.

We now have the following result on the solutions to the dynamics (15):

Theorem 1. *The set* Ψ *in* (14) *is invariant with respect to the dynamics* (15)*. The solutions to the dynamics* (15) *converge to a critical point of the objective functional* μ_2 *in* (13)*. A critical point of* μ_2 *is locally asymptotically stable with respect to the dynamics* (15) *only if it is a strict local minimum.* •

6. NUMERICAL RESULTS

In this section, we present some numerical simulation results that can illustrate the concepts and algorithms of the previous sections.

First, we consider a disk-shaped domain Ω of unit radius, and the placement of a hole B of radius of 0.1 units. Figure 1 shows a plot of μ_2 for the residual domain $\Omega \setminus B$ as a function of h (distance between the center of the disk and the center of the hole). Since the hole is of radius 0.1 units and is contained in Ω, we note that $h \in [0, 0.9)$. Figure 2 shows a contour plot of μ_2 as a function of hole position. We observe from Figures 1 and 2 that the second (also the smallest positive) eigenvalue of the Laplace operator for a disk-shaped domain with a hole increases with the distance between the centers of the domain and the hole, but also appears to decrease as the hole approaches close to the domain boundary (around $h = 0.85$ units). Moreover, μ_2 as a function of h appears to be a convex in the interval $h \in [0, 0.85]$ and concave for $h \in (0.85, 0.9)$. We now consider a ring-shaped domain or annulus with 0.4 unit inner radius and unit outer radius, with the hole radius being 0.1 units. Figure 3 is a plot of μ_2 for the residual domain as a function of h (distance between the center of the ring and the center of the hole). Note that in this case $h \in (0.5, 0.9)$, since the inner radius of the ring is 0.4 units. Figure 4 is a contour plot of μ_2 as a function of hole position. We observe from Figures 3 and 4 that the second eigenvalue of the Laplace operator for a ring-shaped domain or annulus with an hole increases with the distance between the centers of the domain and the hole, but also appears to decrease as the hole approaches close to the outer boundary of the annulus. Moreover, μ_2 as a function of h appears to be concave. We now present simulation results for the projected gradient dynamics (15). For the simulation, we have separated the dynamics into two time scales, with x (the center of the hole) as the slow-scale variable and ψ the fast-scale variable. We consider the case of the disk-shaped domain, that is, the dynamics (15) corresponds to hole placement for the disk-shaped domain to minimize μ_2

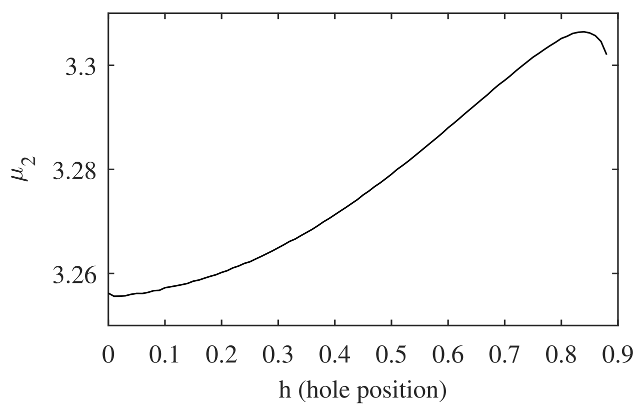

Fig. 1. μ_2 as a function of h for a disk-shaped domain.

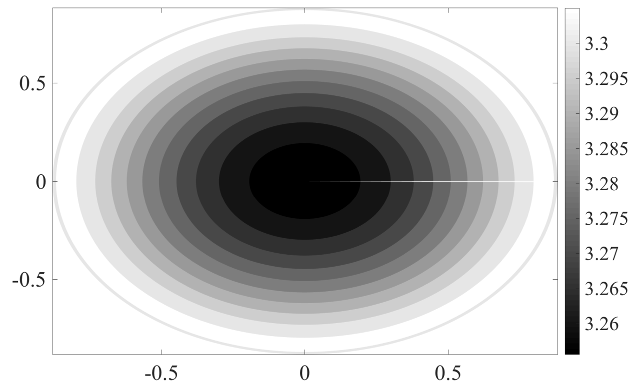

Fig. 2. Contour plot of μ_2 as a function of hole position.

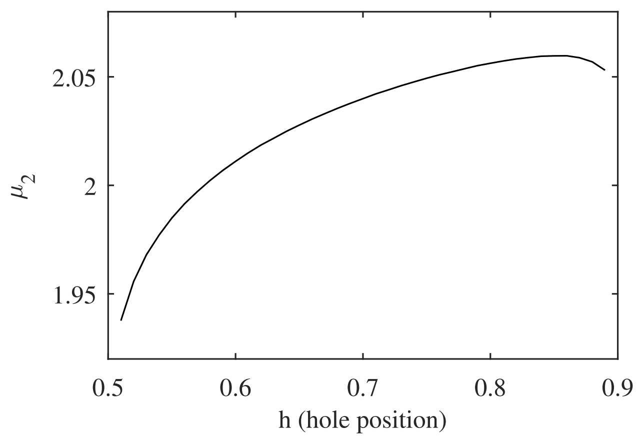

Fig. 3. μ_2 as a function of h for a ring-shaped domain of inner radius 0.4 and unit outer radius.

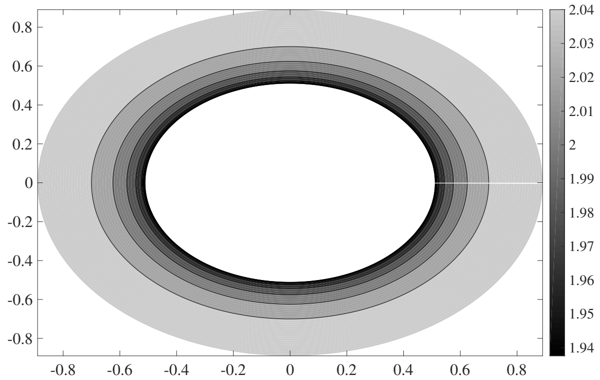

Fig. 4. Contour plot of μ_2 as a function of hole position

of the residual domain. Figure 5 is a plot of $x(t)$, the path of the center of the hole, on the spatial domain, for two different initial conditions $x(0) = (0.4, 0.5)$ and $x(0) = (-0.5, -0.5)$. We observe that the hole center approaches the center of the disk with time, approximately along a straight line.

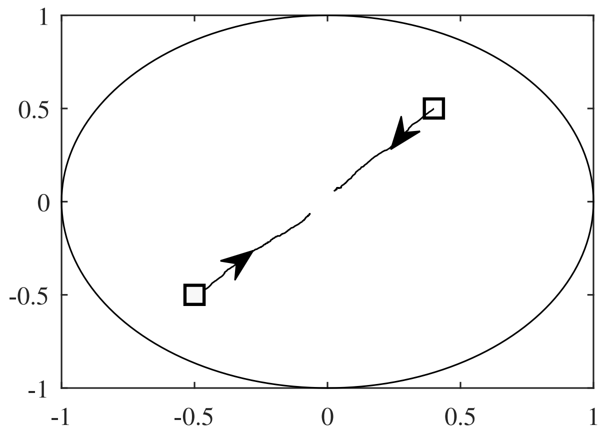

Fig. 5. Path of the center of the hole, $x(t)$ from two different initial conditions $x(0) = (0.4, 0.5)$ and $x(0) = (-0.5, -0.5)$

7. CONCLUSIONS

In this paper, we studied the problem of identifying the critical nodes for consensus in large-scale spatial networks. We began by making a functional approximation of the Laplacian matrix of the graph to the Laplace operator on the domain. In addition to being a natural step in the large-N limit, the real advantage of the approximation is that it does not conceal the geometry of the problem, which is important for spatial networks such as swarms and sensor networks. As a starting point, we looked at the removal of balls of given measure from the domain. The generalization of this setting to arbitrary sets, with a non-uniform distribution of nodes in the domain, is subject of future work. We note that the proposed gradient dynamics were centralized in nature, the problem of designing distributed dynamics for critical node set identification is also of interest and left for future work. Finally, the dependence of the eigenvalue on the hole position needs further analytical exploration for more physical insight into the problem.

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